

# NASH HOMOTOPY SPHERES ARE STANDARD

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ABSTRACT. We prove that the infinite family of homotopy 4-spheres constructed by Daniel Nash are all diffeomorphic to  $S^4$ .

## 0. INTRODUCTION

In [D], D.Nash constructed an infinite family of smooth homotopy 4-spheres  $\Sigma_{p,q,r,s}$  indexed by  $(p, q, r, s) \in \mathbb{Z}^4$ , and conjectured that they are possibly not diffeomorphic to  $S^4$ . Here we prove that they are all diffeomorphic to  $S^4$ . In spirit, Nash's construction is an easier version of the construction of the Akhmedov-Park in [AP], namely one starts with a standard manifold  $X^4 = X_1 \smile_{\partial} X_2$  which is a union of two basic pieces along their boundaries, then does the “log transform” operations to some imbedded tories in both sides with the hope of getting an exotic copy of a known manifold  $M^4$ . In Nash's case  $X$  is the double of  $T_0^2 \times T_0^2$  and  $M$  is  $S^4$ , in Akhmedov-Park case  $X$  is the “Cacime surface” of [CCM] (see [A1] for decomposition of  $X$ ) and  $M$  is  $S^2 \times S^2$ .

## 1. LOG TRANSFORM OPERATION

First we need to recall the *log transform* operation. Let  $X$  be a smooth 4-manifold which contains a torus  $T^2$  with the trivial normal bundle  $\nu(T^2) \approx T^2 \times B^2$ . Let  $\varphi_p$  ( $p \geq 0$ ) be the self-diffeomorphism of  $T^3$  induced by the automorphism

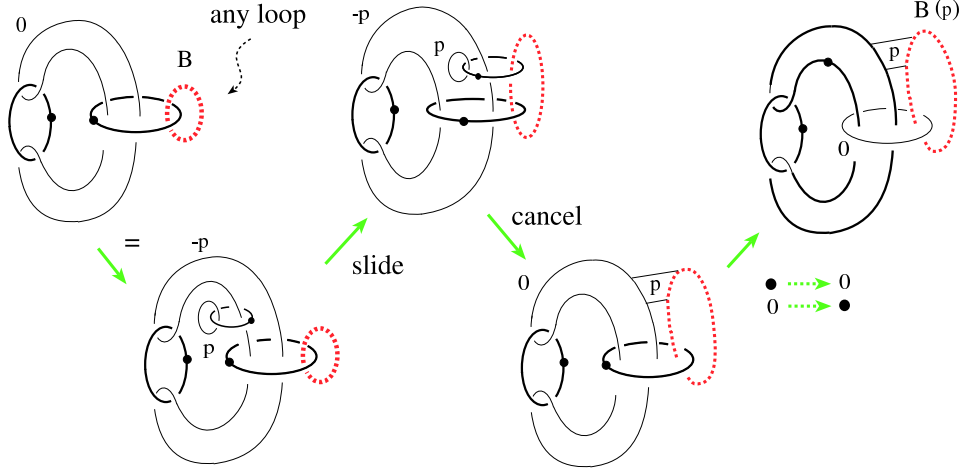
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

of  $H_1(S^1; \mathbf{Z}) \oplus H_1(S^1; \mathbf{Z}) \oplus H_1(S^1; \mathbf{Z})$  with the obvious basis  $(a, b, c)$ . The operation of removing  $\nu(T)$  from  $X$  and then regluing  $T^2 \times B^2$  via  $\varphi_p : S^1 \times T^2 \rightarrow \partial\nu(T)$  is called the  $p$  log-transform of  $X$  along  $T^2$ . In short we will refer this as  $(a \times b, b, p)$  log transform. Figure 1 describes this as a handlebody operation (c.f. [AY] and [GS]).

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FIGURE 1.  $p$  log-transform operation

## 2. HANDLEBODY DESCRIPTIONS OF $T^2 \times T^2$ AND $T_0^2 \times T_0^2$

We follow the recipe of [A1] for drawing the surface bundles over surfaces. Here we have the simpler case of base and fiber are tori. Figure 2 describes the handlebody of  $T^2$  and its thickening  $T^2 \times [0, 1]$ . Figure 3 is a handlebody of  $T^2 \times S^1$ . Note that without the handle  $v$  (the green curve) it is just  $T_0^2 \times S^1$ . So Figure 4 is the handlebody of  $T^2 \times T^2$ . Note that without 2-handles  $u$  and  $v$  (the green and blue curves) Figure 4 it is just  $T_0^2 \times T_0^2$ . By gradually converting the 1-handles of Figure 4 from “pair of balls” notation to the “circle-with-dot” notation of [A2] we get the handlebody pictures of  $T^2 \times T^2$  and  $T_0^2 \times T_0^2$  in Figure 5 (the first and the second pictures). Then by an isotopy (indicated by the arrow) we obtained the first picture of Figure 6 which is  $T_0^2 \times T_0^2$ .

## 3. NASH SPHERES ARE STANDARD

Let  $X_{p,q}$  be the manifold obtained from  $T_0^2 \times T_0^2$  by the log-transformations  $(a \times c, a, p)$  and  $(b \times c, b, q)$ , where  $a, b, c, d$  are the circle factors of  $T_0^2 \times T_0^2$  indicated of Figure 6. Then Nash homotopy spheres are defined to be:

$$\Sigma_{p,q,r,s} = X_{p,q} \smile_{\phi} -X_{r,s}$$

where  $\phi$  is the involution on  $\partial(T_0^2 \times T_0^2)$  flipping  $T_0^2 \times S^1$  and  $S^1 \times T_0^2$ . Notice that if  $X^{r,s}$  is the manifold obtained from  $T_0^2 \times T_0^2$  by the log

transformations  $(c \times a, c, r)$  and  $(d \times a, d, s)$  then we can identify:

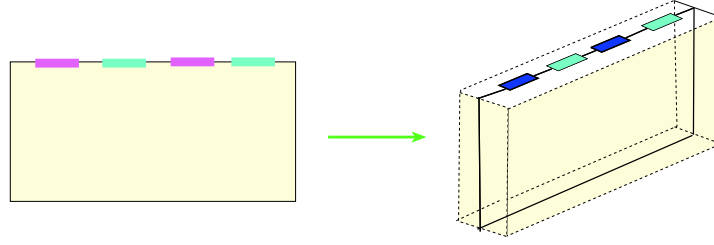
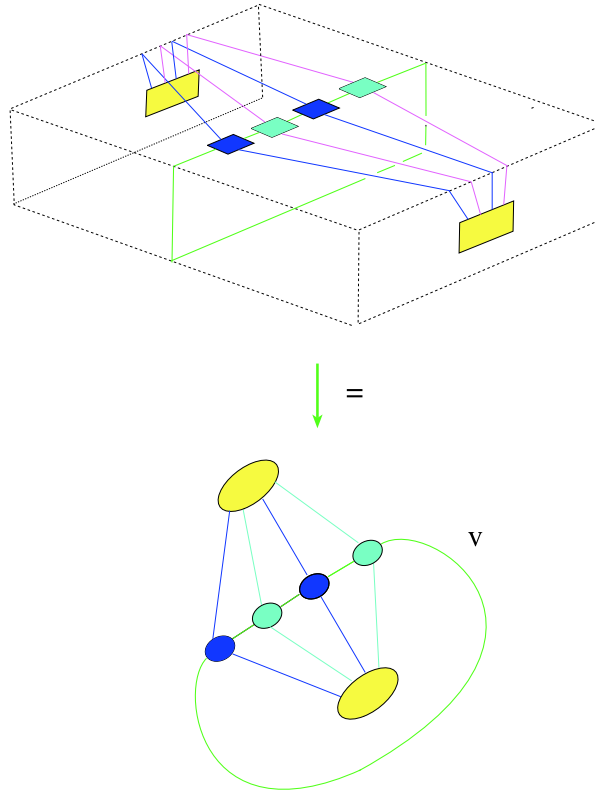
$$\Sigma_{p,q,r,s} = -X_{p,q} \smile_{id} X^{r,s}$$

**Theorem 1.**  $\Sigma_{p,q,r,s} = S^4$

*Proof.* By using the description of the log-transform in Figure 1, we see that the second picture of Figure 6 is  $X_{p,q}$ . Now we will turn  $X_{p,q}$  upside down and glue it to  $X^{r,s}$  along its boundary. For this we take the image of the dual 2-handle curves of  $X_{p,q}$  (indicated by the dotted circles in Figure 6) by the diffeomorphism  $\partial(X_{p,q}) \xrightarrow{\approx} \partial(X^{r,s})$ , and then attach 2-handles to  $X^{r,s}$  along the image of these curves. By reversing the log-transform process of Figure 6 we obtain the first picture of Figure 7, which is just  $T_0^2 \times T_0^2$ , with the dual handle curves indicated! By an isotopy we obtain the second picture of Figure 7, also describing  $T_0^2 \times T_0^2$ . Now again by using the recipe of Figure 1 we perform the  $(c \times a, c, r)$  and  $(d \times a, d, s)$  log-transforms and obtain the first picture of Figure 8, which is  $X^{r,s}$ , with the dual 2-handle curves of  $X_{p,q}$  clearly visible in the picture. Now by attaching 2-handles to top of  $X^{r,s}$  along these curves we obtain  $\Sigma_{p,q,r,s}$ , which is also described by the first picture of Figure 8. Here we should mention our convention: when a framing is not indicated in figures it means the zero framing. Now by the obvious handle slides and cancellations in Figure 8  $\leadsto$  Figure 9 we obtain  $\sharp^4(S^2 \times B^2)$ , and the four 3-handles cancel these to give  $S^4$   $\square$ .

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FIGURE 2.  $T^2 \rightarrow T^2 \times [0, 1]$ FIGURE 3.  $T^2 \times S^1$ 

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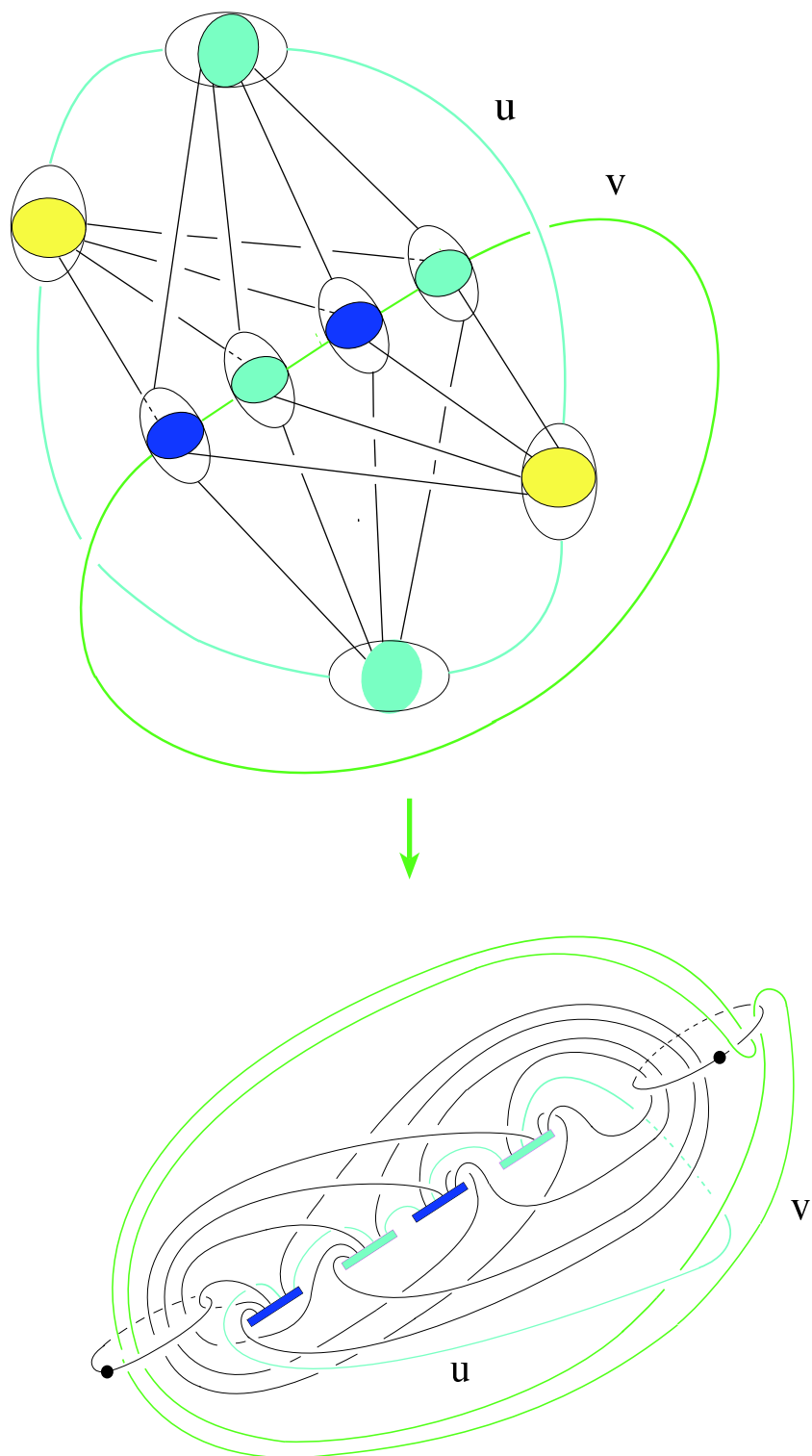


FIGURE 4.  $T^2 \times T^2$

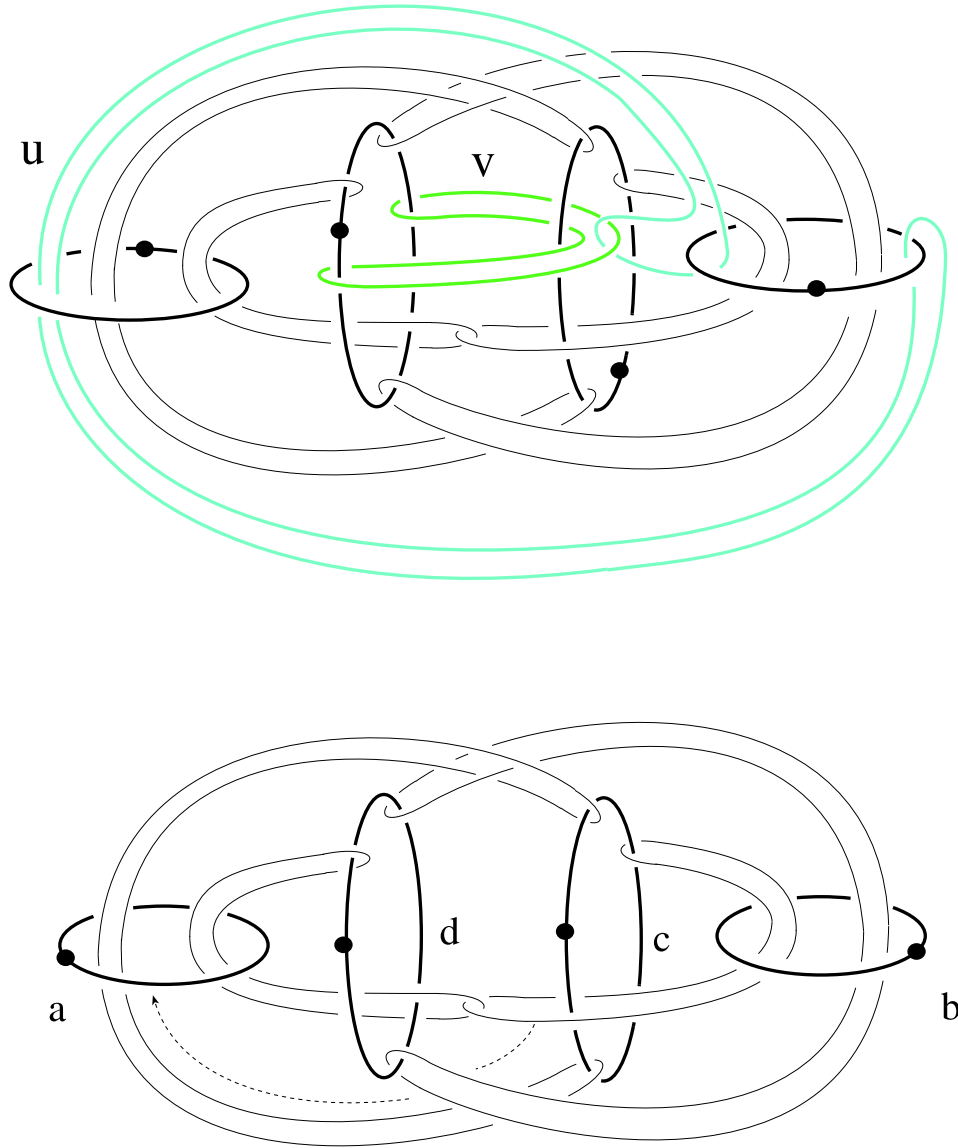


FIGURE 5.  $T^2 \times T^2$  and  $T_0^2 \times T_0^2$

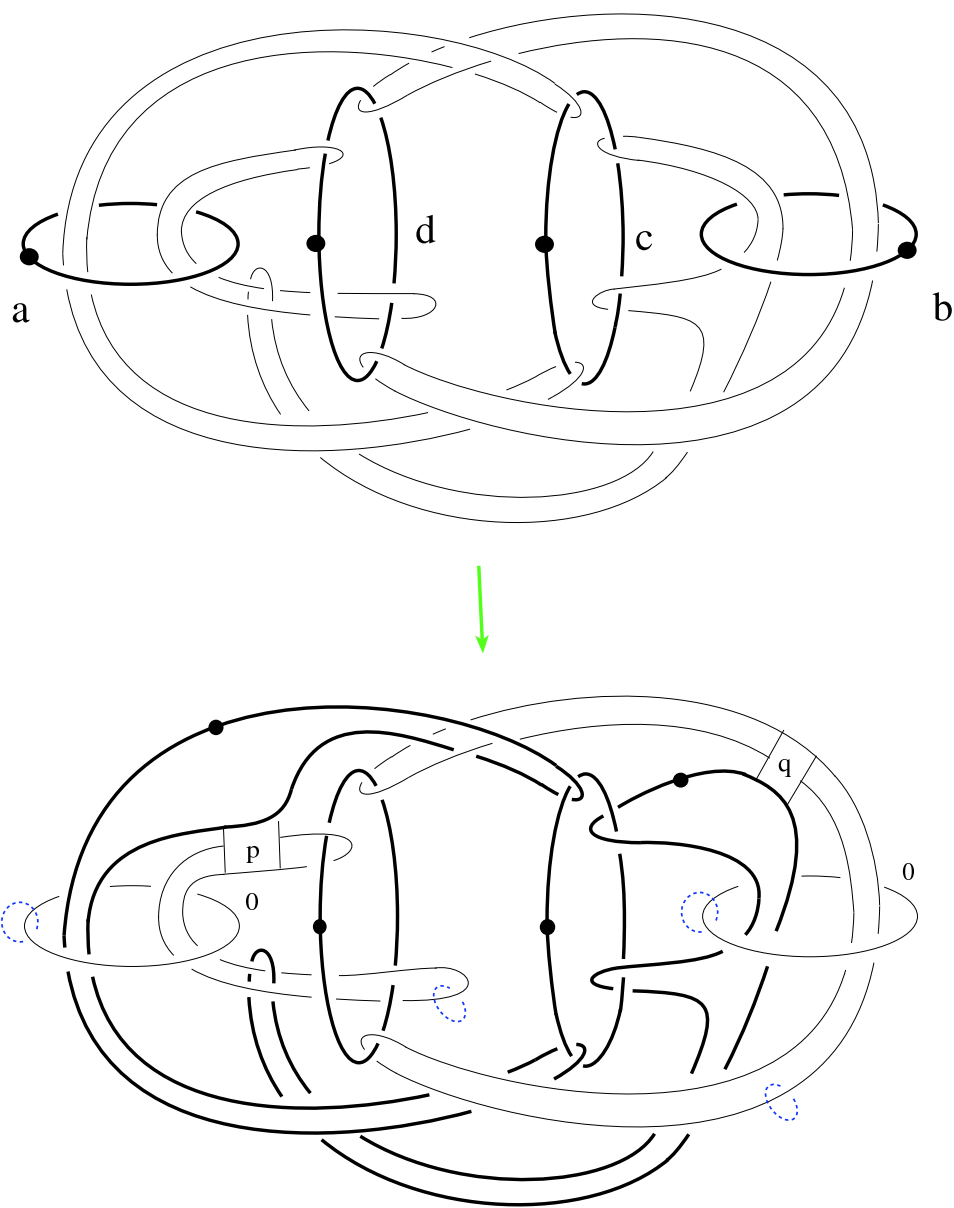


FIGURE 6. log-transformation  $T_0^2 \times T_0^2 \rightarrow X_{p,q}$

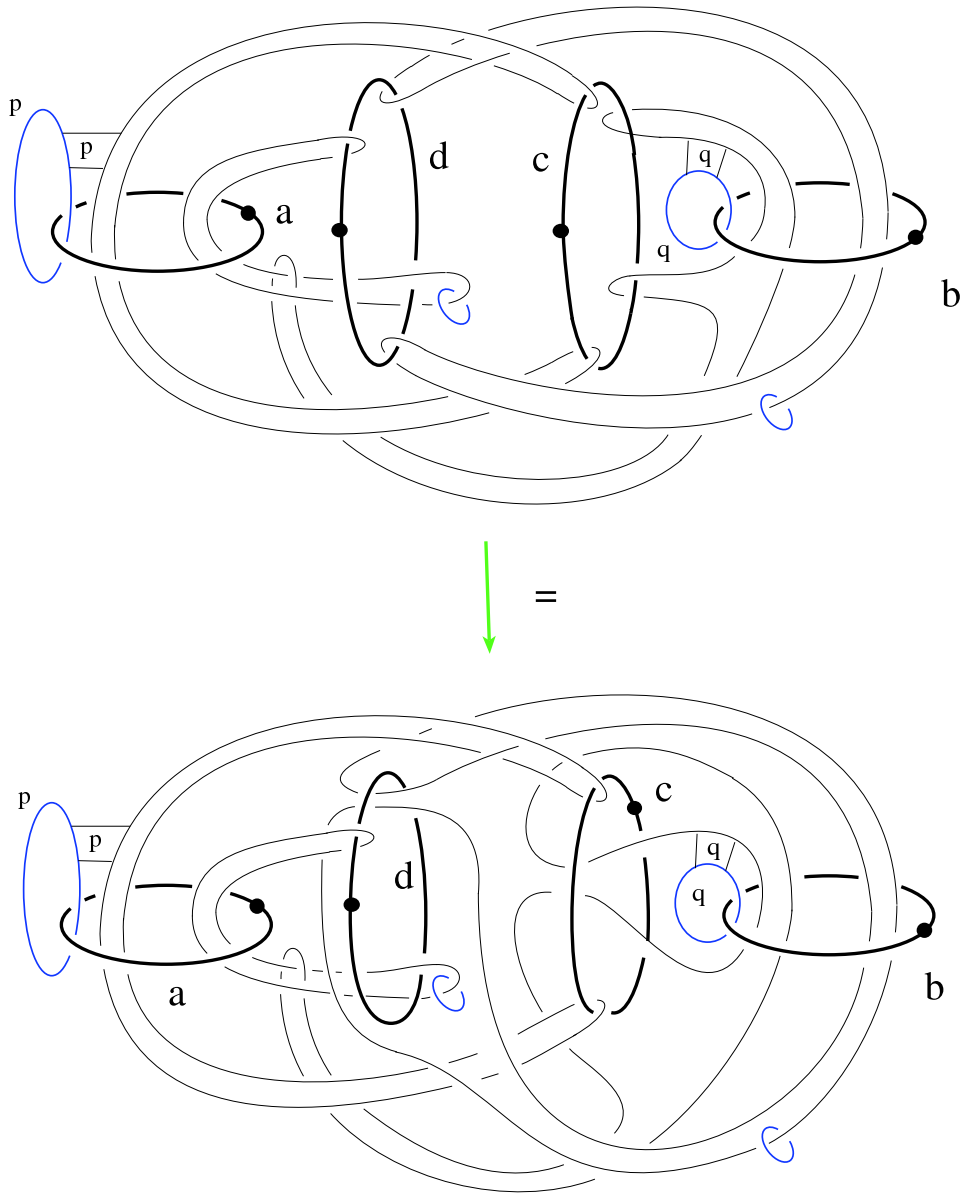


FIGURE 7. log-transformation undone



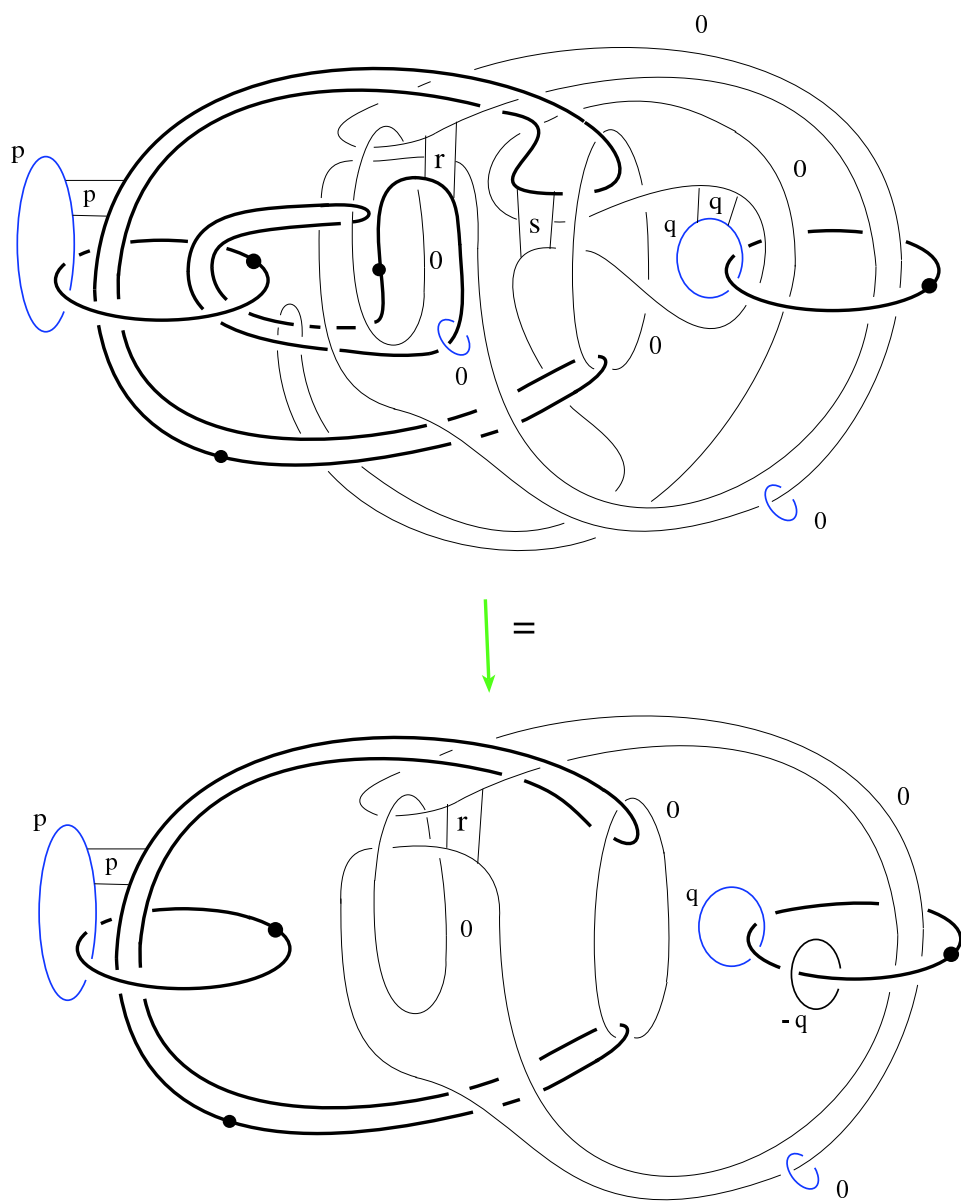


FIGURE 8.  $\Sigma_{p,q,r,s}$

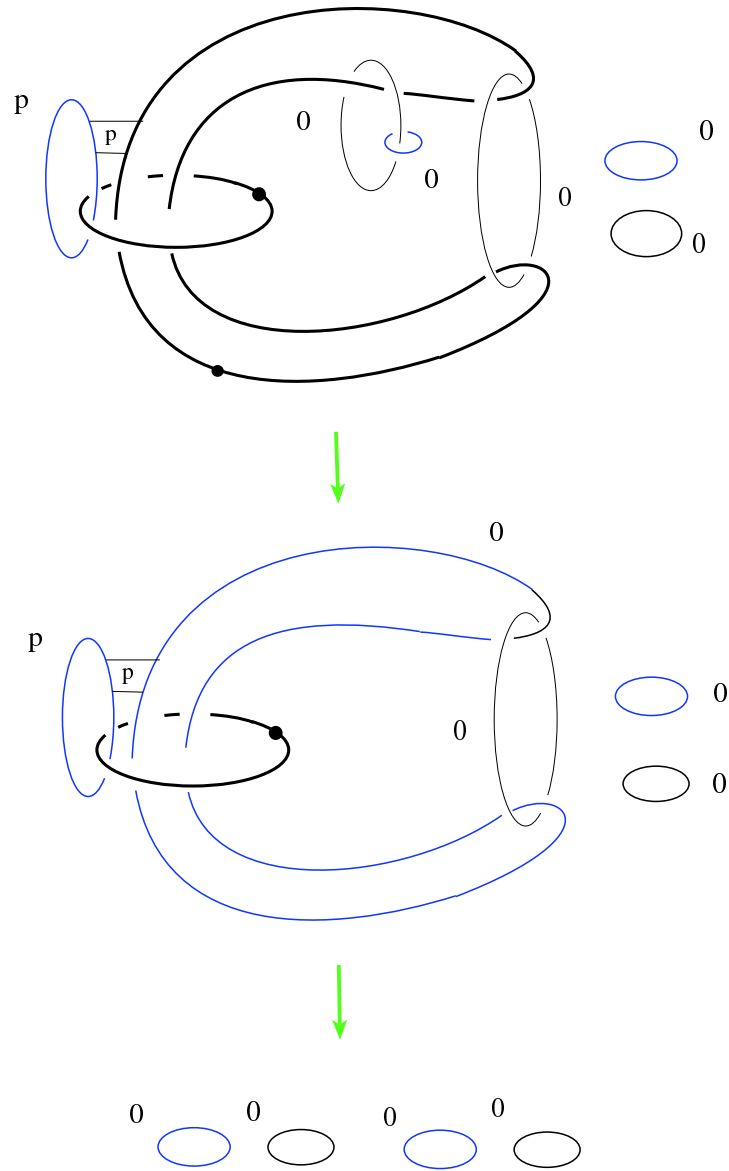


FIGURE 9.